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On a new relation between Jacobi and homogeneous Poisson manifolds

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Abstract
We establish a new, very close, relationship which links Jacobi structures and homogeneous Poisson structures defined on the same manifold and study the characteristic foliations of the related structures. Several examples of this construction are also given.

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1. Introduction

The notion of a Jacobi structure on a manifold, introduced by Lichnerowicz in [14], includes as particular cases important geometric structures, among which are the symplectic, co-symplectic, Poisson, contact and locally conformal symplectic structures, and provides a new framework for the study of these structures. Introducing also the notion of a homogeneous Poisson manifold, Lichnerowicz [14] and Justino [8] set up a very close connection that links these manifolds with the Jacobi manifolds, known as Poissonization. Dazord *et al* [4], investigating this connection in detail, proved that: (1) a 1-codimensional submanifold of a homogeneous Poisson manifold, transverse to the homothety vector field, possesses a Jacobi structure induced by the homogeneous Poisson structure of the manifold and any Jacobi manifold may be obtained in this manner; (2) a 1-codimensional submanifold of a Jacobi manifold, transverse to the Jacobi vector field, possesses a homogeneous Poisson structure induced by the Jacobi structure of the manifold and any homogeneous Poisson manifold may be obtained in this manner.

It is remarkable that the above referred relations between Jacobi and homogeneous Poisson manifolds concern manifolds whose dimensions differ by unity. In this paper, we establish a new relation between these two structures on the same manifold. After a brief review of the basic definitions and results on Jacobi manifolds (section 2), in section 3, we prove that on any homogeneous Poisson manifold we may construct a Jacobi structure

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(proposition 3.4) and, in a converse sense, that on any Jacobi manifold we may build, at least locally, a homogeneous Poisson structure (proposition 3.10). Related questions on these constructions are studied and several examples are also presented. In the final section (section 4), an extensive study of the characteristic foliations of the related Jacobi and homogeneous Poisson manifolds is developed.

Notation. In this paper, M is a C^∞-differentiable manifold of finite dimension n. We denote by TM and T^*M, respectively, the tangent and cotangent bundles over M, $C^\infty(M,\mathbb{R})$ the space of all real C^∞-differentiable functions on M, $\Omega^k(M)$, $0 \leqslant k \leqslant n$, the space of all exterior differentiable k-forms on M, and $\mathcal{V}^k(M)$, $0 \leqslant k \leqslant n$, the space of all skew-symmetric contravariant k-tensor fields on M. Also, $\Omega(M) = \oplus_{k=0}^n \Omega^k(M)$ and $\mathcal{V}(M) = \oplus_{k=0}^n \mathcal{V}^k(M)$ denote, respectively, the algebras of all skew-symmetric covariant and contravariant tensor fields on M.

For the Schouten bracket (cf [13, 24]) and the interior product of a form with a multi-vector field, we use the convention of sign indicated by Koszul [11, 20].

2. Jacobi manifolds

A *Jacobi manifold* (M, Λ, E) is a C^∞-differentiable manifold M of finite dimension endowed with a bivector field Λ and a vector field E such that

$$[\Lambda, \Lambda] = -2E \wedge \Lambda \qquad \text{and} \qquad L_E\Lambda = [E, \Lambda] = 0 \tag{1}$$

where $[,]$ denotes the Schouten bracket [11, 24] and L the Lie derivative operator. We say that (Λ, E) defines a *Jacobi structure* on M.

Defining a Jacobi structure (Λ, E) on M is equivalent to defining an internal composition law $\{,\}_{(\Lambda,E)}$ on $C^\infty(M,\mathbb{R})$

$$\{f, g\}_{(\Lambda,E)} = \Lambda(\mathrm{d}f, \mathrm{d}g) + \langle f\mathrm{d}g - g\mathrm{d}f, E\rangle \qquad f, g \in C^\infty(M,\mathbb{R}) \tag{2}$$

that is bilinear, skew-symmetric, and verifies, for all $f, g, h \in C^\infty(M,\mathbb{R})$, the Jacobi identity

$$\left\{f, \{g, h\}_{(\Lambda,E)}\right\}_{(\Lambda,E)} + \left\{g, \{h, f\}_{(\Lambda,E)}\right\}_{(\Lambda,E)} + \left\{h, \{f, g\}_{(\Lambda,E)}\right\}_{(\Lambda,E)} = 0$$

and the local condition

$$\mathrm{support}\{f, g\}_{(\Lambda,E)} \subseteq (\mathrm{support}\ f) \cap (\mathrm{support}\ g).$$

The bracket $\{,\}_{(\Lambda,E)}$ is called the *Jacobi bracket* associated with (Λ, E) and the space $C^\infty(M,\mathbb{R})$ endowed with the Jacobi bracket (2) is a local Lie algebra in the sense of Kirillov [9]. Conversely, a local Lie algebra structure on $C^\infty(M,\mathbb{R})$ yields a Jacobi structure on M [6, 9].

A Jacobi manifold (M, Λ, E) on which the vector field E identically vanishes is called a *Poisson manifold* [13, 26] and is denoted by (M, Λ). In this case, conditions (1) reduce to

$$[\Lambda, \Lambda] = 0$$

and its associated bracket $\{,\}_\Lambda$ on $C^\infty(M,\mathbb{R})$ is a *Poisson bracket*, i.e. it endows $C^\infty(M,\mathbb{R})$ with a Lie algebra structure and, for all $f, g, h \in C^\infty(M,\mathbb{R})$, the Leibnitz rule holds:

$$\{f, gh\}_\Lambda = \{f, g\}_\Lambda h + g\{f, h\}_\Lambda.$$

We denote by $\Lambda^\#\colon T^*M \to TM$ the vector bundle map associated with Λ, i.e. for all sections α, β of T^*M,

$$\langle \beta, \Lambda^\#(\alpha)\rangle = \Lambda(\alpha, \beta).$$

This map can be seen as a homomorphism of $C^\infty(M,\mathbb{R})$-modules $\Lambda^\#\colon \Omega^1(M) \to \mathcal{V}^1(M)$ and it can be extended to a homomorphism, which we also denote by $\Lambda^\#$, from the space $\Omega^k(M)$ onto the space $\mathcal{V}^k(M)$, $k \in \mathbf{N}$, as follows:

$$\Lambda^\#(f) = f \qquad \text{and} \qquad \Lambda^\#(\sigma)(\alpha_1,\ldots,\alpha_k) = (-1)^k\sigma(\Lambda^\#(\alpha_1),\ldots,\Lambda^\#(\alpha_k)) \tag{3}$$

for all $f \in C^\infty(M,\mathbb{R})$, $\sigma \in \Omega^k(M)$ and $\alpha_1,\ldots,\alpha_k \in \Omega^1(M)$. Finally, we denote by $(\Lambda, E)^\#\colon T^*M \times \mathbb{R} \to TM \times \mathbb{R}$ the vector bundle map associated with (Λ, E), i.e. for all sections α, β of T^*M and for all $f \in C^\infty(M,\mathbb{R})$,

$$(\Lambda, E)^\#(\alpha, f) = (\Lambda^\#(\alpha) + fE, -\langle\alpha, E\rangle).$$

A vector field on (M, Λ, E) of type

$$X_f = \Lambda^\#(\mathrm{d}f) + fE \qquad f \in C^\infty(M,\mathbb{R}) \tag{4}$$

is called the *Hamiltonian vector field associated with* f and the function f is called the *Hamiltonian function of* X_f. We remark that E is the Hamiltonian vector field of (M, Λ, E) associated with the constant function 1.

The image $\mathrm{Im}\Lambda^\#$ of the vector bundle map $\Lambda^\#$ and vector field E define over M a distribution with singularities, called the *characteristic distribution of* (M, Λ, E), which is completely integrable, [3, 6, 9]. Therefore, the maximal integral submanifolds of $\langle \mathrm{Im}\,\Lambda^\#, E\rangle$ form a Stefan foliation of M [25], called the *characteristic foliation* of (M, Λ, E). The leaves of this foliation are called the *characteristic leaves of the Jacobi structure* (Λ, E) of M. When, at a given point $x \in M$, $E(x) \in \mathrm{Im}\,\Lambda^\#(x)$ (respectively $E(x) \notin \mathrm{Im}\,\Lambda^\#(x)$), then the same property holds at all points of the leaf which contains x. That leaf is then of even (respectively odd) dimension. Obviously, the orbits of the Hamiltonian vector fields (4) generate the characteristic leaves of (M, Λ, E). When E identically vanishes on M, i.e. Λ is a Poisson tensor on M, the characteristic foliation of (M, Λ) is its symplectic foliation and the characteristic leaves of (M, Λ) are its symplectic leaves [12].

If the characteristic distribution $\langle \mathrm{Im}\,\Lambda^\#, E\rangle$ of a Jacobi manifold (M, Λ, E) coincides with TM, (M, Λ, E) is said to be *transitive*. According to the parity of the dimension of M, there are two kinds of transitive Jacobi manifolds:

(1) If M has odd dimension, (Λ, E) is defined by a contact 1-form (see [4, 14]).
(2) If M has even dimension, (Λ, E) is defined by a locally conformal symplectic structure (see [4, 14]).

We note that the Jacobi structure of a Jacobi manifold induces a transitive Jacobi structure on each of its characteristic leaves [4, 14].

Let $a \in C^\infty(M,\mathbb{R})$ be a function that never vanishes on (M, Λ, E). We denote by (Λ^a, E^a) the pair formed on M by the bivector field $\Lambda^a := a\Lambda$ and the vector field $E^a := \Lambda^\#(\mathrm{d}a) + aE$. It defines another Jacobi structure on M, which is said to be *a-conformal* to that given initially. Its associated Jacobi bracket on $C^\infty(M,\mathbb{R})$ is given by

$$\{f, g\}_{(\Lambda^a, E^a)} = \frac{1}{a}\{af, ag\}_{(\Lambda, E)} \qquad \forall f, g \in C^\infty(M,\mathbb{R})$$

and, of course, it endows $C^\infty(M,\mathbb{R})$ with a new local Lie algebra structure. The structures (Λ, E) and (Λ^a, E^a) are said to be *conformally equivalent*. The equivalence class of the Jacobi structures on M that are conformally equivalent to a given Jacobi structure is called a *conformal Jacobi structure* on M.

Let (M_1, Λ_1, E_1) and (M_2, Λ_2, E_2) be two Jacobi manifolds and $\phi\colon M_1 \to M_2$ a differentiable map. If (Λ_1, E_1) and (Λ_2, E_2) are ϕ-related, i.e. at each point $x \in M$, $T_x\phi(\Lambda_1(x)) = \Lambda_2(\phi(x))$ and $T_x\phi(E_1(x)) = E_2(\phi(x))$, then $\phi\colon M_1 \to M_2$ is said to be a *Jacobi morphism* or a *Jacobi map*. When $\phi\colon M_1 \to M_2$ is a diffeomorphism, the Jacobi structures (Λ_1, E_1) and (Λ_2, E_2) are said to be *equivalent*.

A map $\phi\colon (M_1, \Lambda_1, E_1) \to (M_2, \Lambda_2, E_2)$ is called an *a-conformal Jacobi map* if there exists a function $a \in C^\infty(M_1, \mathbb{R})$ that never vanishes on M_1 such that $\phi\colon \left(M_1, \Lambda_1^a, E_1^a\right) \to (M_2, \Lambda_2, E_2)$ is a Jacobi map.

A vector field X on a Jacobi manifold (M, Λ, E) is said to be an *infinitesimal Jacobi automorphism* (respectively a *conformal infinitesimal Jacobi automorphism*) of (Λ, E) if its flow defines Jacobi automorphisms (respectively conformal Jacobi automorphisms) on M. This fact is equivalent to $L_X\Lambda = [X, \Lambda] = 0$ and $L_X E = [X, E] = 0$ (respectively $L_X\Lambda = [X, \Lambda] = a\Lambda$ and $L_X E = [X, E] = \Lambda^\#(\mathrm{d}a) + aE$, for a function $a \in C^\infty(M, \mathbb{R})$) [8].

For further details and expositions see, e.g., [14, 18, 19].

3. Homogeneous Poisson manifolds and Jacobi manifolds

Definition 3.1. *A homogeneous Poisson manifold (M, Λ, T) is a Poisson manifold (M, Λ) equipped with a vector field T, called the homothety vector field, such that*

$$L_T\Lambda = [T, \Lambda] = -\Lambda.$$

The particular close relationships that exist between homogeneous Poisson manifolds and Jacobi manifolds were already indicated and studied extensively in [4]. Precisely, Dazord *et al* [4] have shown the following propositions.

Proposition 3.2 [4]**.** *Let (M, Λ, T) be a homogeneous Poisson manifold and Σ a submanifold of M, of codimension 1, transverse to the homothety vector field T. Then, Σ has an induced Jacobi structure $(\Lambda_\Sigma, E_\Sigma)$ characterized by one of the following properties:*

1. *For any pair (f, g) of homogeneous functions of degree 1 with respect to T, defined on an open subset $\mathcal{O}$ of M, the Jacobi bracket of f and g, restricted to $\Sigma \cap \mathcal{O}$, is the restriction of the Poisson bracket of f and g to $\Sigma \cap \mathcal{O}$.*
2. *Let $\pi\colon U \to \Sigma$ be the projection on Σ of a tubular neighbourhood U of Σ in M such that, for any $x \in \Sigma$, $\pi^{-1}(x)$ is a connected arc of the integral curve of T through x. Let a be a function on U, equal to 1 on Σ and homogeneous of degree 1 with respect to T. Then, the projection π is an a-conformal Jacobi map.*

Proposition 3.3 [4]**.** *Let (M, Λ, E) be a Jacobi manifold and N a 1-codimensional submanifold of M transverse to E. Let $\pi\colon U \to N$ be the projection on N of a tubular neighbourhood U of N in M such that, for any $x \in N$, $\pi^{-1}(x)$ is a connected arc of the integral curve of E through x. Let η be the 1-form along N that verifies $i(E)\eta = 1$ and $i(X)\eta = 0$, for any vector field X on M tangent to N. Then, there exists on N a unique Poisson structure such that π is a Jacobi map; this structure, which is homogeneous with respect to the homothety vector field $\Lambda^\#(\eta)$, is called the homogeneous Poisson structure induced on N by the Jacobi structure of M.*

On the other hand, it is well known that with any Jacobi manifold (M, Λ, E) we may associate a homogeneous Poisson manifold $(\tilde{M}, \tilde{\Lambda}, \tilde{T})$ by setting

$$\tilde{M} = M \times \mathbb{R} \qquad \tilde{\Lambda} = \mathrm{e}^{-t}\left(\Lambda + \frac{\partial}{\partial t} \wedge E\right) \qquad \text{and} \qquad \tilde{T} = \frac{\partial}{\partial t} \tag{5}$$

where t is the canonical coordinate on the factor $\mathbb{R}$, see [8, 14]. The manifold $(\tilde{M}, \tilde{\Lambda}, \tilde{T})$ so defined is called the *Poissonization* of the Jacobi manifold (M, Λ, E).

We observe that the relations between homogeneous Poisson manifolds and Jacobi manifolds mentioned above concern manifolds whose dimensions differ by unity. In the following proposition, we establish another close relation between these two structures defined on the same manifold.

Proposition 3.4. *Let (M, Λ, T) be a homogeneous Poisson manifold and E a vector field on M such that*

$$[E, \Lambda - T \wedge E] = 0. \tag{6}$$

Then, the pair (C, E), where

$$C := \Lambda - T \wedge E, \tag{7}$$

defines a Jacobi structure on M.

Proof. By construction, the second condition of (1) holds. We compute

$$\begin{aligned}[C, C] &= [\Lambda - T \wedge E, \Lambda - T \wedge E]\\ &= [\Lambda, \Lambda] - 2[\Lambda, T \wedge E] + [T \wedge E, T \wedge E]\\ &= -2[\Lambda, T] \wedge E + 2T \wedge [\Lambda, E] - 2[T, E] \wedge E \wedge T\\ &= -2\Lambda \wedge E - 2T \wedge ([E, \Lambda - T \wedge E]) \overset{(6)(7)}{=} -2E \wedge C.\end{aligned}$$

Hence, (C, E) endows M with a Jacobi structure. □

Remark 3.5. Given a homogeneous Poisson manifold (M, Λ, T) it is natural to ask about the existence of a vector field E on M verifying (6). The answer is that such a vector field always exists locally. For example, E may be a Hamiltonian vector field with respect to Λ whose Hamiltonian function f is a homogeneous function with respect to T, i.e. $L_T f = \lambda f$, $\lambda \in \mathbb{R}$. Effectively, in this case, $[E, T] = [\Lambda^{\#}(\mathrm{d}f), T] = -[[\Lambda, f], T] = [\Lambda, [T, f]] + [f, [T, \Lambda]] = [\Lambda, \lambda f] - [f, \Lambda] = (1 - \lambda)\Lambda^{\#}(\mathrm{d}f) = (1 - \lambda)E$. Hence, $[E, \Lambda - T \wedge E] = 0$.

Remark 3.6. It is easy to check that the Poisson bracket $\{,\}_\Lambda$ and the Jacobi bracket $\{,\}_{(C,E)}$ on $C^\infty(M, \mathbb{R})$ coincide on the vector subspace of $C^\infty(M, \mathbb{R})$ formed by the homogeneous functions of degree 1 with respect to T, i.e. the functions $f \in C^\infty(M, \mathbb{R})$ that verify $L_T f = f$.

Remark 3.7. Under the assumption that the space $\hat{M} = M/E$ of the integral curves of E has the structure of a manifold for which the canonical projection $\pi\colon M \to \hat{M}$ is a submersion, condition (6) assures that both the bivector fields Λ and $C = \Lambda - T \wedge E$ are reduced via (M, E) [21] to the same bivector field $\hat{\Lambda}$ on $\hat{M}$, i.e. $\hat{\Lambda} = \pi_*(\Lambda) = \pi_*(C)$, which is a Poisson tensor. Hence, $\pi\colon M \to \hat{M}$ is simultaneously a Poisson and a Jacobi map. Moreover, if T is a projectable vector field, its projection $\hat{T} = \pi_*(T)$ is a homothety vector field of $\hat{\Lambda}$ [22].

The relationship between the Poissonization (5) of the constructed Jacobi structure (C, E) on M and the initially given Poisson structure Λ is studied in the next proposition.

Proposition 3.8. *Under the same hypothesis and notations as in proposition 3.4, let $(\tilde{M}, \tilde{\Lambda}, \tilde{T})$ be the Poissonization of the Jacobi structure (C, E) and $\tilde{\pi}\colon \tilde{M} \to M$ the projection of $\tilde{M}$ on M parallel to the integral curves of $\partial/\partial t - T$. Then, $\tilde{\pi}\colon (\tilde{M}, \tilde{\Lambda}) \to (M, \Lambda)$ is a Poisson map.*

Proof. We identify M with the submanifold $M \times \{0\}$ of $\tilde{M} = M \times \mathbb{R}$ and remark that the vector field $\partial/\partial t - T$ is transverse to M. Hence, M may be also seen as the set of the integral curves of $\partial/\partial t - T$. We consider the projection $\tilde{\pi}\colon \tilde{M} \to M$ of $\tilde{M}$ on M parallel to the integral curves of $\partial/\partial t - T$ that maps each point (x, t) of $\tilde{M} = M \times \mathbb{R}$ to the unique point x' of M such that $(x', 0)$ and (x, t) belong to the same integral curve of $\partial/\partial t - T$. Since

$$\left[\frac{\partial}{\partial t} - T, \tilde{\Lambda}\right] = -\mathrm{e}^{-t}\left(\frac{\partial}{\partial t} - T\right) \wedge (E + [T, E])$$

we have that $\tilde{\Lambda} = \mathrm{e}^{-t}(\Lambda + (\partial/\partial t - T) \wedge E)$ is projectable by $\tilde{\pi}$ and its projection is $\tilde{\pi}_*(\tilde{\Lambda}) = \Lambda$. □

In the following, we present some examples concerning the result of proposition 3.4.

Examples 3.9

1. *Jacobi structures on vector spaces*: First, we consider an orientable manifold M of dimension n with a volume element ν and recall the following construction of the operator D: $\mathcal{V}(M) \to \mathcal{V}(M)$ due to Koszul [11].

Given ν, it induces an isomorphism Φ: $\mathcal{V}^k(M) \to \Omega^{n-k}(M), 0 \leqslant k \leqslant n$, defined by $\Phi(Q) = i(Q)\nu$, $Q \in \mathcal{V}^k(M)$, where i denotes the interior product of a form with a multivector field on M. We introduce the operator D: $\mathcal{V}(M) \to \mathcal{V}(M)$

$$D := -\Phi^{-1} \circ \mathrm{d} \circ \Phi$$

(d being the exterior derivative of the differential forms). D is of degree -1 and of square 0, it generates the Schouten bracket on $\mathcal{V}(M)$, i.e. for $P \in \mathcal{V}^p(M)$ and $Q \in \mathcal{V}^q(M)$,

$$[P, Q] = (-1)^p(D(P \wedge Q) - D(P) \wedge Q - (-1)^p P \wedge D(Q)) \tag{8}$$

and it is a derivation of the Schouten bracket [10, 11]. For a vector field X on M, $D(X) = -\mathrm{div}_\nu(X)$, where div_ν denotes the divergence with respect to ν, and for a Poisson tensor Λ on M, $D(\Lambda)$ is its modular vector field that verifies $[D(\Lambda), \Lambda] = 0$ (see [10, 11] and references therein).

Now, we assume that $M = V$ is a vector space. Let Λ be a Poisson bivector on V whose components are homogeneous polynomials of degree k, i.e. if $(x_1, \ldots, x_n)$ are linear coordinates on V,

$$\Lambda_{ij}(x) = \sum_{i_1,\ldots,i_s=1}^{n} c_{ij}^{i_1\cdots i_s} x_{i_1}^{n_{i_1}} \cdots x_{i_s}^{n_{i_s}} \qquad (x \in V)$$

with $n_{i_1} + \cdots + n_{i_s} = k$ (the quantities $c_{ij}^{i_1\cdots i_s}$ are constants, $c_{ij}^{i_1\cdots i_s} = -c_{ji}^{i_1\cdots i_s}$ and $c_{ij}^{i_1\cdots i_m\cdots i_r\cdots i_s} = c_{ij}^{i_1\cdots i_r\cdots i_m\cdots i_s}$), and let $T = \sum_{i=1}^n x_i\partial/\partial x_i$ be the radial vector field on V. Then, Λ has a unique decomposition $\Lambda = C + T \wedge E$, where C and E are, respectively, the D-exact homogeneous 2-tensor field and D-exact homogeneous vector field [17]. From (C, E) we derive a Jacobi structure on V. Effectively, we have $[T, \Lambda] = (k-2)\Lambda$. But

$$[T, \Lambda] \overset{(8)}{=} -(D(T \wedge \Lambda) - D(T) \wedge \Lambda + T \wedge D(\Lambda)).$$

Hence,

$$(k-2)\Lambda = -D(T \wedge \Lambda) - n\Lambda - T \wedge D(\Lambda)$$

and

$$\Lambda = -\frac{1}{n+k-2}\,(D(T \wedge \Lambda) + T \wedge D(\Lambda)). \tag{9}$$

By putting

$$C = -\frac{1}{n+k-2} D(T \wedge \Lambda) \qquad \text{and} \qquad E = -\frac{1}{n+k-2} D(\Lambda)$$

we obtain the decomposition of Λ mentioned above. A simple computation yields

$$[E, C] = 0 \qquad \text{and} \qquad [C, C] = -2(2-k)E \wedge C$$

(because E is the modular vector field of Λ and its components are homogeneous polynomials of degree $k-1$). Thus, the pair $(C, (2-k)E)$ defines a Jacobi structure on V.

We remark that:

(i) If Λ is a linear Poisson bivector, i.e. $k = 1$ (in this case, it is well known that the constants c_{ij}^m, $i, j, m = 1, \ldots, n$, are the structural constants of a Lie algebra structure on the dual space V^* of V and Λ is the Lie–Poisson structure on the dual V of the Lie algebra V^* [12, 20]), E is a constant vector field on V. Precisely, E is equal to the linear 1-form $-\frac{1}{n-1}\mathrm{tr}(ad)$ on V^*, where $\mathrm{tr}(ad)$: $u \in V^* \mapsto \mathrm{tr}(ad_u)$ and tr denotes the trace [11]. The condition $[E, C] = 0$ is equivalent to the 1-cocycle condition for E.

(ii) If Λ is a quadratic Poisson bivector, i.e. $k = 2$, then its associated Jacobi structure $(C, (2-k)E) = (C, 0)$ is a Poisson structure; it is exactly the one that appears in the canonical decomposition of quadratic Poisson structures established in [15].

2. *Linear Jacobi structures on vector bundles*: We present the construction of a linear Jacobi structure on the dual A^* to a Lie algebroid $(A, [\![,]\!], \varrho)$ over a differentiable manifold M given in [7]. This structure may be viewed as a Jacobi structure associated with the homogeneous linear Poisson structure on A^*, in the sense of proposition 3.4.

A *Lie algebroid* $(A, [\![,]\!], \varrho)$ over a differentiable manifold M is a vector bundle $\pi \colon A \to M$ endowed with a Lie algebra structure $[\![,]\!]$ on its space $\Gamma(A)$ of the global cross sections and with an *anchor morphism* $\varrho \colon A \to TM$ of vector bundles such that, if we also denote by $\varrho \colon \Gamma(A) \to \mathcal{V}^1(M)$ the homomorphism of $C^\infty(M, \mathbb{R})$-modules induced by the anchor morphism, then, for every $s_1, s_2 \in \Gamma(A)$ and $f \in C^\infty(M, \mathbb{R})$,

$$\varrho([\![s_1, s_2]\!]) = [\varrho(s_1), \varrho(s_2)] \qquad \text{and} \qquad [\![s_1, f s_2]\!] = f[\![s_1, s_2]\!] + \left(L_{\varrho(s_1)} f\right) s_2.$$

We choose coordinates $(x_1, \ldots, x_n)$ on an open neighbourhood U of M and a local basis of sections $(e_1, \ldots, e_r)$ of $\pi \colon A \to M$ in U. With respect to this choice, the bracket $[\![,]\!]$ and the anchor morphism ϱ are determined by structure functions $c_{ij}^k, \varrho_i^l \in C^\infty(U, \mathbb{R})$, as

$$[\![e_i, e_j]\!] = \sum_{k=1}^{r} c_{ij}^k e_k \qquad \varrho(e_i) = \sum_{l=1}^{n} \varrho_i^l \frac{\partial}{\partial x_l}.$$

Let $(x_1, \ldots, x_n, \mu_1, \ldots, \mu_r)$ be the induced linear coordinates on the dual bundle A^*, i.e. $\mu_i = \langle \cdot, e_i \rangle$, $i = 1, \ldots, r$. By setting

$$\Lambda = \sum_{1 \leqslant i < j \leqslant r} \sum_{k=1}^{r} c_{ij}^k \mu_k \frac{\partial}{\partial \mu_i} \wedge \frac{\partial}{\partial \mu_j} + \sum_{i=1}^{r} \sum_{l=1}^{n} \varrho_i^l \frac{\partial}{\partial \mu_i} \wedge \frac{\partial}{\partial x_l} \qquad \text{and} \qquad T = \sum_{i=1}^{r} \mu_i \frac{\partial}{\partial \mu_i}$$

we define a homogeneous Poisson structure on A^* such that the Poisson bracket of linear functions is again linear [2].

Next, we introduce the Lie algebroid cohomology complex with trivial coefficients [16] whose space of all 1-cocycles is the set of the sections $\varphi \in \Gamma(A^*)$ that verify

$$\langle \varphi, [\![s_1, s_2]\!] \rangle = L_{\varrho(s_1)} \langle \varphi, s_2 \rangle - L_{\varrho(s_2)} \langle \varphi, s_1 \rangle \qquad \forall s_1, s_2 \in \Gamma(A).$$

We have that, if $\varphi \in \Gamma(A^*)$, $\varphi = \sum_{i=1}^{r} \varphi_i e^i$ with $\varphi_i \in C^\infty(U, \mathbb{R})$ and $(e^1, \ldots, e^r)$ the dual basis of $(e_1, \ldots, e_r)$, is a 1-cocycle, then its vertical lift [5] $\varphi^v = \sum_{i=1}^{r} \varphi_i \partial/\partial \mu_i$ satisfies

$$[\Lambda, \varphi^v] = 0 \qquad \text{and} \qquad [\varphi^v, T] = \varphi^v.$$

Therefore, the pair (C, E), where

$$C = \Lambda - T \wedge \varphi^v \qquad \text{and} \qquad E = \varphi^v,$$

is a Jacobi structure on A^*. The fact that the Jacobi bracket $\{,\}_{(C,E)}$ of linear functions on A^* is again linear [7] is an immediate consequence of remark 3.6 and the analogous result for $\{,\}_\Lambda$.

In particular, when M is a point, A is a Lie algebra $\mathcal{G}$ and Λ is the Lie–Poisson structure on $\mathcal{G}^* = A^*$. By taking $\varphi = \mathrm{tr}(ad) \in \mathcal{G}^* = \Gamma(\mathcal{G}^*)$, $\mathrm{tr}(ad) : x \in \mathcal{G} \mapsto \mathrm{tr}(ad_x)$ where tr denotes the trace, φ^v is the modular vector field of Λ and the obtained Jacobi structure on $\mathcal{G}^* = A^*$ coincides with the Jacobi structure of example 3.9.1 (for $k = 1$).

For further examples of linear Jacobi structures on vector bundles, see [7].

3. *Jacobi structures on the cotangent bundle of a Lie group*: Let G be a Lie group of dimension n with Lie algebra $(\mathcal{G}, [,])$. We equip the cotangent bundle T^*G of G with the canonical symplectic Poisson structure Λ, i.e. Λ is defined by the inverse of the differential of the Liouville 1-form on T^*G [12].

In order to facilitate certain calculations, we trivialize T^*G and identify it with $\mathcal{G}^* \times G$, via the right trivialization of T^*G (see [1, 12]). We fix a basis $(X_1, \ldots, X_n)$ of the Lie algebra $\mathcal{G}$ with structure constants c_{ij}^k, i.e. $[X_i, X_j] = \sum_{k=1}^n c_{ij}^k X_k$, and consider the dual basis $(\xi_1, \ldots, \xi_n)$ in $\mathcal{G}^*$. Let $(x_1, \ldots, x_n)$ be the associated linear coordinate functions on $\mathcal{G}^*$. In these coordinates the symplectic Poisson structure on $\mathcal{G}^* \times G$ induced by Λ via the right trivialization of T^*G, also denoted by Λ, is given by (see [1])

$$\Lambda = \sum_{i=1}^n R_i \wedge \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i,j,k=1}^n c_{ij}^k x_k \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$$

where R_i, $i = 1, \ldots, n$, are the right invariant vector fields on G which take the values X_i at the neutral element e of G. We observe that Λ is a homogeneous Poisson structure on $\mathcal{G}^* \times G$ with respect to the vector field $T = \sum_{i=1}^n x_i \partial/\partial x_i$. Let E be a left invariant vector field on G. Since, for any right invariant vector field R on G, $[E, R] = 0$ [12], we have that $[E, \Lambda - T \wedge E] = 0$. Thus, the pair $(C, E) = (\Lambda - T \wedge E, E)$ defines a Jacobi structure on $\mathcal{G}^* \times G$. The image of (C, E) by means of the inverse map of the right trivialization of T^*G endows T^*G with a Jacobi structure.

A natural question which arises from the above study is: does any Jacobi structure on M comes from a homogeneous Poisson structure on M, in the sense of proposition 3.4? The answer is:

Proposition 3.10. *Any Jacobi structure (C, E) on a differentiable manifold M may be seen, at least locally, as a Jacobi structure associated with a homogeneous Poisson structure (Λ, T) on M, in the sense of proposition 3.4.*

Proof. Let (C, E) be a Jacobi structure on M. For each point $p \in M$ such that $E(p) \neq 0$, there is an open neighbourhood U of p in M and a function $f \in C^\infty(U, \mathbb{R})$ such that $L_E f = 1$. We take the Hamiltonian vector field $T = C^\#(\mathrm{d}f) + fE$ associated with f. It is a conformal infinitesimal Jacobi automorphism of (C, E) whose conformity function is $a = -L_E f = -1$ [8], i.e. $[T, C] = -C$ and $[T, E] = -E$. We set

$$\Lambda := C + T \wedge E$$

and easily verify that $[\Lambda, \Lambda] = 0$. Obviously, T is a homothety vector field of Λ. Consequently, (Λ, T) defines a homogeneous Poisson structure on U and the restriction of the initially given Jacobi structure (C, E) to U can be considered as the Jacobi structure associated with (Λ, T, E) in the sense of proposition 3.4. □

Remark 3.11.

1. The function f considered in the above proof is a homogeneous function of degree 1 with respect to $T = C^\#(\mathrm{d}f) + fE$ and a Casimir function of $\Lambda = C + T \wedge E = C + C^\#(\mathrm{d}f) \wedge E$. The vector field $C^\#(\mathrm{d}f)$ is also a homothety vector field of Λ. These remarks imply that E is an infinitesimal Poisson automorphism of the constructed Poisson structure Λ.
2. Since $L_E f = 1$ and $L_E \mathrm{d}f = 0$, at each point $x \in U$, $\mathrm{d}f(x)$ generates a 1-dimensional subspace of T_x^*U which is complementary to the annihilator $\langle E(x)\rangle^0$ of the subspace $\langle E(x)\rangle$ of T_xU generated by $E(x)$. So, $T^*U = \langle E\rangle^0 \oplus \langle \mathrm{d}f\rangle$ and by duality $TU = \ker \mathrm{d}f \oplus \langle E\rangle$. The vector sub-bundles $\ker \mathrm{d}f$ and $\langle E\rangle$ of TU being involutive, they define two complementary foliations of U. Then, U can be identified with a product $\hat{M} \times I$ of two manifolds; $\hat{M}$ is interpreted by the set of the integral curves of E and I by the set of the leaves of $\ker \mathrm{d}f$. From remark 3.7 and the fact that f is a Casimir function of Λ, we get that Λ can be identified with the projection of C by $\pi : U \to \hat{M}$. On the other hand, by identifying $\hat{M}$ with a 1-codimensional submanifold of U transverse to E, via a section of π: $U \to \hat{M}$, $(\Lambda, C^\#(\mathrm{d}f))$ can be seen as the homogeneous Poisson structure induced on $\hat{M}$ by the Jacobi structure $(C|_U, E|_U)$ of U (see proposition 3.3).

Some characteristic examples of Jacobi structures and their associated homogeneous Poisson structures on the same manifold are developed in the following.

Example 3.12.

1. *Contact manifolds*: Let M be a $(2n+1)$-dimensional differentiable manifold endowed with a *contact form* ϑ, i.e. ϑ is a 1-form on M such that $\vartheta \wedge (\mathrm{d}\vartheta)^n \neq 0$ holds everywhere on M. We consider on M the Reeb vector field E [23], which is defined by

$$i(E)\vartheta = 1 \qquad \text{and} \qquad i(E)\,\mathrm{d}\vartheta = 0$$

and the bivector field C whose associated vector bundle map $C^\#$: $T^*M \to TM$ is defined, for all sections α of T^*M, by

$$i(C^\#(\alpha))\vartheta = 0 \qquad \text{and} \qquad i(C^\#(\alpha))\,\mathrm{d}\vartheta = -(\alpha - \langle \alpha, E\rangle \vartheta). \tag{10}$$

Then, (C, E) defines a transitive Jacobi structure on M, determined by the contact form ϑ (see [12, 14]).

We observe that $\mathrm{d}\vartheta$ is a presymplectic form of rank $2n$ on M and $\ker \mathrm{d}\vartheta$ is a one-dimensional distribution over M spanned by E. Let f be a function on an open neighbourhood U of a point $p \in M$ such that $L_E f = 1$. Then, developing the same argumentation as in remark 3.11.2, we have $T^*U = \langle E\rangle^0 \oplus \langle \mathrm{d}f\rangle$ and by duality $TU = \ker \mathrm{d}f \oplus \langle E\rangle$. Since $\ker \mathrm{d}\vartheta = \langle E\rangle$, $\mathrm{d}\vartheta$ is a section of $\bigwedge^2 \langle E\rangle^0$ and defines an isomorphism $\mathrm{d}\vartheta^\flat$: $\ker \mathrm{d}f \to \langle E\rangle^0$ given, for all $X \in \ker \mathrm{d}f$, by $\mathrm{d}\vartheta^\flat(X) = -i(X)\,\mathrm{d}\vartheta$. Thus, $\mathrm{d}\vartheta$ endows each integral manifold of the involutive distribution $\ker \mathrm{d}f$ with a symplectic structure. Hence, we obtain a foliation of U by symplectic manifolds. Let Λ be the Poisson structure on U whose symplectic foliation coincides with that of the above. We have that $\ker \Lambda^\# = \langle \mathrm{d}f\rangle$ and, for all semi-basic forms β on U (i.e. the forms $\beta \in \Omega^1(U)$ such that $i(E)\beta = 0$), $i(\Lambda^\#(\beta))\,\mathrm{d}\vartheta = -\beta$. So, for all $\alpha \in \Omega^1(U)$,

$$i(\Lambda^\#(\alpha - \langle \alpha, E\rangle \vartheta))\,\mathrm{d}\vartheta = -(\alpha - \langle \alpha, E\rangle \vartheta)$$

because $\alpha - \langle \alpha, E\rangle \vartheta$ is a semi-basic form on U. Taking into account the second condition of (10) and the above remark, we obtain that, for all $\alpha \in \Omega^1(U)$,

$$i(C^\#(\alpha))\,\mathrm{d}\vartheta = i(\Lambda^\#(\alpha - \langle \alpha, E\rangle \vartheta))\,\mathrm{d}\vartheta$$

which means that $C^{\#}(\alpha) - \Lambda^{\#}(\alpha - \langle\alpha, E\rangle\vartheta) \in \ker \mathrm{d}\vartheta$, i.e. there exists $g \in C^{\infty}(U, \mathbb{R})$ such that

$$C^{\#}(\alpha) - \Lambda^{\#}(\alpha - \langle\alpha, E\rangle\vartheta) = gE.$$

Precisely,

$$g = \langle\vartheta, C^{\#}(\alpha) - \Lambda^{\#}(\alpha - \langle\alpha, E\rangle\vartheta)\rangle \overset{(10)}{=} \langle\alpha, \Lambda^{\#}(\vartheta)\rangle.$$

Consequently, for all $\alpha \in \Omega^1(U)$,

$$C^{\#}(\alpha) = \Lambda^{\#}(\alpha) - \langle\alpha, E\rangle\Lambda^{\#}(\vartheta) + \langle\alpha, \Lambda^{\#}(\vartheta)\rangle E. \tag{11}$$

By setting $T = -\Lambda^{\#}(\vartheta)$, from (11) we get

$$C = \Lambda - T \wedge E.$$

Moreover, T is a homothety vector field of Λ. Effectively,

$$[T, \Lambda] = [-\Lambda^{\#}(\vartheta), \Lambda] = -\partial_{\Lambda}(\Lambda^{\#}(\vartheta)) = -\Lambda^{\#}(\mathrm{d}\vartheta),$$

where ∂_{Λ} denotes the operator of the Lichnerowicz–Poisson cohomology of (U, Λ) [13] and $\Lambda^{\#}\colon \Omega^k(U) \to \mathcal{V}^k(U)$, $k \in \mathbf{N}$, is the homomorphism of $C^{\infty}(U, \mathbb{R})$-modules determined by (3). A simple computation yields $\Lambda^{\#}(\mathrm{d}\vartheta) = \Lambda$. Hence, we conclude that, at least locally, the Jacobi structure (C, E) comes from the homogeneous Poisson structure (Λ, T), in the sense of proposition 3.4.

According to Darboux's theorem [12], there exist canonical coordinates $(x_0, \dots, x_{2n})$ on U such that

$$\vartheta = \mathrm{d}x_0 + \sum_{k=1}^{n} x_{2k-1}\,\mathrm{d}x_{2k} \qquad \text{and} \qquad E = \frac{\partial}{\partial x_0}.$$

Then,

$$\Lambda = \sum_{k=1}^{n} \frac{\partial}{\partial x_{2k-1}} \wedge \frac{\partial}{\partial x_{2k}} \qquad T = \sum_{k=1}^{n} x_{2k-1}\frac{\partial}{\partial x_{2k-1}}$$

$$\text{and} \qquad C = \sum_{k=1}^{n} \frac{\partial}{\partial x_{2k-1}} \wedge \left(\frac{\partial}{\partial x_{2k}} - x_{2k-1}\frac{\partial}{\partial x_0}\right).$$

2. *Locally conformal symplectic manifolds*: Let (M, Ω, ω) be a *locally conformal symplectic manifold*, that is, M is a $2n$-dimensional manifold equipped with a non-degenerate 2-form Ω and a closed 1-form ω, called the *Lee 1-form*, such that

$$\mathrm{d}\Omega + \omega \wedge \Omega = 0.$$

For any $f \in C^{\infty}(M, \mathbb{R})$, the associated Hamiltonian vector field X_f is given by

$$i(X_f)\Omega = -(\mathrm{d}f + f\omega).$$

Let E be the unique vector field and C the unique bivector field on M which are defined by

$$i(E)\Omega = -\omega \qquad \text{and} \qquad i(C^{\#}(\alpha))\Omega = -\alpha, \qquad \text{for all} \qquad \alpha \in \Omega^1(M). \tag{12}$$

Then, (C, E) endows M with a transitive Jacobi structure [6]. Also denoting by $C^{\#}$ the extension (3) of the isomorphism $C^{\#}\colon \Omega^1(M) \to \mathcal{V}^1(M)$ of $C^{\infty}(M, \mathbb{R})$-modules defined by (12), we have that

$$E = C^{\#}(\omega) \qquad C = C^{\#}(\Omega) \qquad \text{and} \qquad X_f = C^{\#}(\mathrm{d}f + f\omega) \qquad f \in C^{\infty}(M, \mathbb{R}).$$

Let f be a function on an open neighbourhood U of a point $p \in M$, $\omega(p) \neq 0$, such that $i(X_f)\omega = -1$. We set

$$\sigma = \Omega + (\mathrm{d}f + f\omega) \wedge \omega$$

and consider the bivector field $\Lambda = C^{\#}(\sigma)$. Then, the pair $(\Lambda, T) = (C^{\#}(\sigma), X_f)$ defines a homogeneous Poisson structure on U and the restriction of the Jacobi structure (C, E) on U can be seen as a Jacobi structure associated with (Λ, T), in the sense of proposition 3.4.

4. Characteristic foliations of (Λ, T) and (C, E)

In this paragraph, and always in the context of proposition 3.4, we study the position of the characteristic leaves of the Jacobi structure (C, E), given by (7), with respect to the characteristic (symplectic) leaves of Λ and the extended characteristic leaves of (Λ, T). First, we recall the notion of an *extended characteristic leaf* of a homogeneous Poisson manifold (M, Λ, T) introduced in [4].

The *extended characteristic distribution* over a homogeneous Poisson manifold (M, Λ, T) is the distribution over M that is generated by the image $\mathrm{Im}\,\Lambda^\#$ of $\Lambda^\#: T^*M \to TM$ and by the homothety vector field T of Λ. It is completely integrable [4] and defines a Stefan foliation [25] of M, called the *extended characteristic foliation* of (M, Λ, T). The leaves of this foliation, denoted S^{ext}, are called the *extended characteristic leaves* of (M, Λ, T). If an extended characteristic leaf S^{ext} of (M, Λ, T) is of even dimension, it is a symplectic leaf S of (M, Λ, T), T is tangent to S and its restriction $T|_S$ is a homothety vector field of the symplectic Poisson structure of S. If S^{ext} is of odd dimension $2k+1$, it is foliated by symplectic leaves of (M, Λ, T), all of dimension $2k$. In this case, the vector field $T|_{S^{\mathrm{ext}}}$ is transverse to these symplectic leaves and its flow ψ maps these symplectic leaves, one to the other, by conformal symplectic transformations, i.e. the pull-back of the symplectic form of a symplectic leaf by ψ is proportional to the symplectic form of another leaf.

Under the assumptions of proposition 3.4, we have

$$|\mathrm{rank}\,\Lambda(x) - \mathrm{rank}\,(T \wedge E)(x)| \leqslant \mathrm{rank}\,C(x) \leqslant \mathrm{rank}\,\Lambda(x) + \mathrm{rank}\,(T \wedge E)(x) \qquad x \in M \tag{13}$$

and also

$$C^\# = \Lambda^\# - \langle\cdot, T\rangle E + \langle\cdot, E\rangle T \Leftrightarrow C^\# + \langle\cdot, T\rangle E = \Lambda^\# + \langle\cdot, E\rangle T$$

which means that the characteristic distribution $\langle \mathrm{Im}\,C^\#, E\rangle$ of (M, C, E) and the extended characteristic distribution $\langle \mathrm{Im}\Lambda^\#, T\rangle$ of (M, Λ, T) have a common subdistribution $F = \mathrm{Im}(C^\# + \langle\cdot, T\rangle E) = \mathrm{Im}(\Lambda^\# + \langle\cdot, E\rangle T)$. Hence, the position of the characteristic leaves of (M, C, E) with respect to the symplectic leaves of (M, Λ) and the extended characteristic leaves of (M, Λ, T) depends on the position of the vector fields T and E with respect to $\mathrm{Im}\,\Lambda^\#$.

We consider an open neighbourhood U of a point in M, restrict the tensor fields Λ, T, E and C to U, and distinguish the following cases:

1. $T \in \mathrm{Im}\,\Lambda^\#$ and $E \in \mathrm{Im}\,\Lambda^\#$ on U. Then $\mathrm{Im}\,\Lambda^\# = \langle \mathrm{Im}\,\Lambda^\#, T\rangle$, $F \subseteq \mathrm{Im}\,\Lambda^\#$, $\mathrm{Im}\,C^\# \subseteq \mathrm{Im}\,\Lambda^\#$ and

$$F \subseteq \langle \mathrm{Im}\,C^\#, E\rangle \subseteq \mathrm{Im}\,\Lambda^\#. \tag{14}$$

Let τ and ε be two sections of T^*U such that $T = \Lambda^\#(\tau)$ and $E = \Lambda^\#(\varepsilon)$.

- If $\langle\tau, E\rangle = f \neq -1 \Leftrightarrow \langle\varepsilon, T\rangle = -f \neq 1$, $f \in C^\infty(U, \mathbb{R})$, we have that $T \in F$, because $T = \Lambda^\#((1+f)^{-1}\tau) + \langle(1+f)^{-1}\tau, E\rangle T$, hence $\mathrm{Im}\,\Lambda^\# \subseteq F$ and $E = C^\#((1+f)^{-1}\varepsilon) \in \mathrm{Im}\,C^\#$. Consequently, $F = \mathrm{Im}\,\Lambda^\#$, $\langle \mathrm{Im}\,C^\#, E\rangle = \mathrm{Im}\,C^\#$ and, from (14), we get $\langle \mathrm{Im}\,C^\#, E\rangle = \mathrm{Im}\,\Lambda^\#$. Therefore, in this case, the characteristic foliation of (U, C, E) coincides with the extended characteristic foliation of (U, Λ, T) which is its symplectic foliation.
- If $\langle\tau, E\rangle = -1 \Leftrightarrow \langle\varepsilon, T\rangle = 1$, we have $C^\#(\tau) = 0$ and $C^\#(\varepsilon) = 0$. So $\tau, \varepsilon \in (\mathrm{Im}\,C^\#)^0$, where $(\mathrm{Im}\,C^\#)^0$ denotes the annihilator of $\mathrm{Im}\,C^\#$, $E \notin \mathrm{Im}\,C^\#$ (because, if $E \in \mathrm{Im}\,C^\#$, $\langle\tau, E\rangle = 0$) and $T \notin \langle \mathrm{Im}\,C^\#, E\rangle$ (because, if $T \in \langle \mathrm{Im}\,C^\#, E\rangle$, $\langle\varepsilon, T\rangle = 0$). Thus, at each $x \in U$, $\mathrm{rank}\,C(x) = \mathrm{rank}\,\Lambda(x) - 2$, and the characteristic leaves of (U, C, E) are of odd dimension and transverse to T. Taking into account (14), we conclude that each

$2k$-dimensional symplectic leaf S of (U, Λ, T), which is also an extended characteristic leaf, is foliated by $(2k-1)$-dimensional characteristic leaves $\mathcal{C}$ of (U, C, E) transverse to T. The transitive Jacobi structure of each $\mathcal{C}$ coincides with the Jacobi structure induced on $\mathcal{C}$, seen as a 1-codimensional submanifold of S transverse to T, by the homogeneous symplectic Poisson structure of S, in the sense of proposition 3.2.

2. $T \in \mathrm{Im}\,\Lambda^{\#}$ and $E \notin \mathrm{Im}\,\Lambda^{\#}$ on U. Then $F \subseteq \mathrm{Im}\,\Lambda^{\#} = \langle \mathrm{Im}\,\Lambda^{\#}, T\rangle$ and, at each $x \in U$, $\mathrm{rank} C(x) = \mathrm{rank}\Lambda(x)$, which gives

$$\dim \mathrm{Im}\, C^{\#} = \dim \mathrm{Im}\,\Lambda^{\#} \tag{15}$$

on U. Also, $T \in F$, because $T = \Lambda^{\#}(\alpha) + \langle \alpha, E\rangle T$, where α is a section of the annihilator $(\mathrm{Im}\,\Lambda^{\#})^0$ of $\mathrm{Im}\,\Lambda^{\#}$ such that $\langle \alpha, E\rangle = 1$. This fact implies $F = \mathrm{Im}\,\Lambda^{\#} = \langle \mathrm{Im}\,\Lambda^{\#}, T\rangle$ and $T = C^{\#}(\alpha) \in \mathrm{Im}\, C^{\#}$. Moreover, $E \notin \mathrm{Im}\, C^{\#}$. Effectively, if $E \in \mathrm{Im}\, C^{\#}$, $\langle \mathrm{Im}\, C^{\#}, E\rangle = \mathrm{Im}\, C^{\#}$; but $\mathrm{Im}\,\Lambda^{\#} = F \subseteq \langle \mathrm{Im}\, C^{\#}, E\rangle$; hence $\mathrm{Im}\,\Lambda^{\#} \subseteq \mathrm{Im}\, C^{\#} = \langle \mathrm{Im}\, C^{\#}, E\rangle$ and, since (15) holds, $\mathrm{Im}\,\Lambda^{\#} = \langle \mathrm{Im}\, C^{\#}, E\rangle$. The latter implies $E \in \mathrm{Im}\,\Lambda^{\#}$ on U, which is in contradiction with our assumption. Consequently, in this case, the extended characteristic foliation of (U, Λ, T) coincides with its symplectic foliation and the characteristic leaves of (U, C, E) are of odd dimension. Any $(2k+1)$-dimensional characteristic leaf $\mathcal{C}$ of (U, C, E) is foliated by $2k$-dimensional symplectic leaves S of (U, Λ, T) transverse to E. The homogeneous symplectic Poisson structure of each S coincides with the homogeneous Poisson structure induced on S, considered as a 1-codimensional submanifold of $\mathcal{C}$ transverse to E, by the transitive Jacobi structure of $\mathcal{C}$, in the sense of proposition 3.3.

3. $T \notin \mathrm{Im}\,\Lambda^{\#}$ and $E \in \mathrm{Im}\,\Lambda^{\#}$ on U. Then: (i) $E \in \mathrm{Im}\, C^{\#}$, since $E = C^{\#}(\beta)$, where β is a section of the annihilator $(\mathrm{Im}\,\Lambda^{\#})^0$ of $\mathrm{Im}\,\Lambda^{\#}$ such that $\langle \beta, T\rangle = -1$, so,

$$F \subseteq \langle \mathrm{Im}\, C^{\#}, E\rangle = \mathrm{Im}\, C^{\#} \tag{16}$$

and (ii) at each $x \in U$, $\mathrm{rank}\, C(x) = \mathrm{rank}\,\Lambda(x)$, hence (15) also holds. On the other hand, we have that $\ker \Lambda^{\#} = \ker(\Lambda^{\#} + \langle \cdot, E\rangle T)$, because $\ker \Lambda^{\#} \subseteq \ker(\Lambda^{\#} + \langle \cdot, E\rangle T)$ and there is no $\xi \in \ker(\Lambda^{\#} + \langle \cdot, E\rangle T)$ such that $\xi \notin \ker \Lambda^{\#}$ (if there was such ξ, we would have $\Lambda^{\#}(\xi) + \langle \xi, E\rangle T = 0$, which gives: (i) if $\langle \xi, E\rangle = 0$, $\Lambda^{\#}(\xi) = 0$, a result in contradiction with the assumption $\xi \notin \ker \Lambda^{\#}$, and (ii) if $\langle \xi, E\rangle \neq 0$, $T = -\langle \xi, E\rangle^{-1}\Lambda^{\#}(\xi) \in \mathrm{Im}\,\Lambda^{\#}$, a result in contradiction with the assumption $T \notin \mathrm{Im}\,\Lambda^{\#}$ on U). Thus, $\dim F = \dim \mathrm{Im}\,(\Lambda^{\#} + \langle \cdot, E\rangle T) = \dim \mathrm{Im}\,\Lambda^{\#}$ on U. Taking into account (15) and the latter relation, (16) yields $F = \mathrm{Im}\, C^{\#} = \langle \mathrm{Im}\, C^{\#}, E\rangle$ on U. Also, we have $F = \mathrm{Im}\,(\Lambda^{\#} + \langle \cdot, E\rangle T) \subset \langle \mathrm{Im}\,\Lambda^{\#}, T\rangle$ on U. These facts mean that, at each point $x \in U$, the characteristic leaf $\mathcal{C}$ of (U, C, E) through x intersects transversely the symplectic leaf S of (U, Λ) through x, their intersection contains the integral curve of E passing by x, and both leaves have the same even dimension. Also, each $(2k+1)$-dimensional extended characteristic leaf S^{ext} of (U, Λ, T) is foliated simultaneously by $2k$-dimensional characteristic leaves $\mathcal{C}$ of (U, C, E) and $2k$-dimensional symplectic leaves S of (U, Λ). Both foliations of S^{ext} are transverse to T. The transitive Jacobi structure of each $\mathcal{C}$ is the structure induced on $\mathcal{C}$, seen as a 1-codimensional submanifold of S^{ext} transverse to T, by the homogeneous Poisson structure of S^{ext}, in the sense of proposition 3.2.

4. $T \notin \mathrm{Im}\,\Lambda^{\#}$ and $E \notin \mathrm{Im}\,\Lambda^{\#}$ on U. Then $T \in F$, since $T = \Lambda^{\#}(\alpha) + \langle \alpha, E\rangle T$, where $\alpha \in (\mathrm{Im}\,\Lambda^{\#})^0$ and $\langle \alpha, E\rangle = 1$. Hence, $\mathrm{Im}\,\Lambda^{\#} \subset F \subseteq \langle \mathrm{Im}\,\Lambda^{\#}, T\rangle$ on U, which implies $F = \langle \mathrm{Im}\,\Lambda^{\#}, T\rangle$ on U, because $\dim \langle \mathrm{Im}\,\Lambda^{\#}, T\rangle = \dim \mathrm{Im}\,\Lambda^{\#} + 1$.

- If there exists a pair (γ, δ) of 1-forms of $(\mathrm{Im}\,\Lambda^{\#})^0$ verifying $C(\gamma, \delta) \neq 0$, we have $E = (C(\gamma, \delta))^{-1}(\langle \delta, E\rangle C^{\#}(\gamma) - \langle \gamma, E\rangle C^{\#}(\delta))$ and $T = (C(\gamma, \delta))^{-1}(\langle \delta, T\rangle C^{\#}(\gamma) - \langle \gamma, T\rangle C^{\#}(\delta))$, i.e. $E, T \in \mathrm{Im}\, C^{\#}$ on U. So, $\mathrm{Im}\, C^{\#} = \langle \mathrm{Im}\, C^{\#}, E\rangle$ and

$$\dim \mathrm{Im}\, C^{\#} = \dim \mathrm{Im}\,\Lambda^{\#} + 2 \tag{17}$$

on U. Also, $F \subset \langle \mathrm{Im}\, C^{\#}, E\rangle$, because $E \notin F$ (if $E \in F$, $\mathrm{Im}\, C^{\#} \subseteq F$ and $\dim \mathrm{Im}\, C^{\#} \leqslant \dim F = \dim \mathrm{Im}\, \Lambda^{\#} + 1$, a result in contradiction with (17)). Consequently, each $(2k+2)$-dimensional characteristic leaf $\mathcal{C}$ of (U, C, E) is foliated by $(2k+1)$-dimensional extended characteristic leaves S^{ext} of (U, Λ, T) transverse to E and each S^{ext} is foliated by $2k$-dimensional symplectic leaves S of (U, Λ) transverse to T. The homogeneous Poisson structure of each S^{ext} is the structure induced on S^{ext}, viewed as a 1-codimensional submanifold of $\mathcal{C}$ transverse to E, by the transitive Jacobi structure of $\mathcal{C}$, in the sense of proposition 3.3.

- If, for every pair (γ, δ) of 1-forms of $(\mathrm{Im}\, \Lambda^{\#})^0$, $C(\gamma, \delta) = 0$ (this case always arises when, at each $x \in U$, $\mathrm{corank}\, \Lambda(x) = 1$), we have that $C^{\#}(\gamma)$ and $C^{\#}(\delta)$ are collinear and they are contained in the plane generated by T and E. Also, $E \notin \mathrm{Im}\, C^{\#}$ on U. Effectively, if $E \in \mathrm{Im}\, C^{\#}$ on U, $\langle \mathrm{Im}\, C^{\#}, E\rangle = \mathrm{Im}\, C^{\#}$ and $\mathrm{Im}\, C^{\#}$ is of even dimension greater than $\dim \mathrm{Im}\, \Lambda^{\#} + 1$ on U, since $F \subseteq \langle \mathrm{Im}\, C^{\#}, E\rangle$. Taking into account (13), the only possibility is $\dim \mathrm{Im}\, C^{\#} = \dim \mathrm{Im}\, \Lambda^{\#} + 2$ on U, which implies $\dim(\mathrm{Im}\, C^{\#})^0 = \dim(\mathrm{Im}\, \Lambda^{\#})^0 - 2$ on U, i.e. there exists $(\gamma, \delta) \in (\mathrm{Im}\, \Lambda^{\#})^0 \times (\mathrm{Im}\, \Lambda^{\#})^0$ such that $C(\gamma, \delta) \neq 0$, a result in contradiction with our assumption. Thus, $\langle \mathrm{Im}\, C^{\#}, E\rangle$ is of odd dimension equal to $\dim \mathrm{Im}\, \Lambda^{\#} + 1$ on U. Consequently, $F = \langle \mathrm{Im}\, \Lambda^{\#}, T\rangle = \langle \mathrm{Im}\, C^{\#}, E\rangle$ on U, which means that the extended characteristic foliation of (U, Λ, T) coincides with the characteristic foliation of (U, C, E). Each $(2k+1)$-dimensional characteristic leaf $\mathcal{C}$ of (U, C, E), which is also an extended characteristic leaf S^{ext} of (U, Λ, T), is foliated by $2k$-dimensional symplectic leaves S of (U, Λ, T) transverse to E and T. The homogeneous symplectic Poisson structure of each S coincides with the structure induced on S, seen as a 1-codimensional submanifold of $\mathcal{C}$ transverse to E, by the transitive Jacobi structure of $\mathcal{C}$, in the sense of proposition 3.3.

Remark 4.1. We note that our study on the characteristic foliations of (Λ, T) and (C, E) does not cover the singular points, i.e. the points $x \in M$ such that $E(x) \in \mathrm{Im}\, \Lambda^{\#}(x)$, or $T(x) \in \mathrm{Im}\, \Lambda^{\#}(x)$, and that any neighbourhood of x contains points y with $E(y) \notin \mathrm{Im}\, \Lambda^{\#}(y)$, or $T(y) \notin \mathrm{Im}\, \Lambda^{\#}(y)$. This study will be the subject of further research.

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References

[1] Alekseevsky D *et al* 1994 Poisson structures on the cotangent bundle of a Lie group or a principle bundle and their reductions *J. Math. Phys.* **35** 4909–27
[2] Courant Th 1990 Dirac manifolds *Trans. Am. Math. Soc.* **319** 631–61
[3] Dazord P 1985 Feuilletages à singularités *Indagationes Math.* **47** 21–39
[4] Dazord P, Lichnerowicz A and Marle Ch-M 1991 Structure locale des variétés de Jacobi *J. Math. Pure Appl.* **70** 101–52
[5] Grabowski J and Urbański P 1995 Tangent lifts of Poisson and related structures *J. Phys. A: Math. Gen.* **28** 6743–77
[6] Guédira F and Lichnerowicz A 1984 Géométrie des algèbres de Lie de Kirillov *J. Math. Pure Appl.* **63** 407–84
[7] Iglesias D and Marrero J C 2000 Some linear Jacobi structures on vector bundles *C. R. Acad. Sci., Paris* I **331** 125–30
[8] Justino A M 1984 Géométrie des variétés de Poisson et des variétés de Jacobi *Thèse de Troisième Cycle* Université Pierre et Marie Curie

[9] Kirillov A 1976 Local Lie algebras *Russ. Math. Surv.* **31** 55–75

[10] Kosmann-Schwarzbach Y 2000 Modular vector fields and Batalin–Vilkovisky algebras *Poisson Geometry (Warsaw 1998) (Banach Center Publications vol 51)* (Warsaw: Polish Academy of Science) pp 109–29

[11] Koszul J-L 1985 Crochet de Schouten–Nijenhuis et cohomologie *Élie Cartan et les Mathématiques d'aujourd'hui (Astérisque, Numéro Hors Série)* pp 257–71

[12] Libermann P and Marle Ch-M 1987 *Symplectic Geometry and Analytical Mechanics* (Dordrecht: Reidel)

[13] Lichnerowicz A 1977 Les variétés de Poisson et leurs algèbres de Lie associées *J. Differ. Geom.* **12** 253–300

[14] Lichnerowicz A 1978 Les variétés de Jacobi et leurs algèbres de Lie associées *J. Math. Pures Appl.* **57** 453–88

[15] Liu Z-J and Xu P 1992 On quadratic Poisson structures *Lett. Math. Phys.* **26** 33–42

[16] Mackenzie K 1987 *Lie Groupoids and Lie Algebroids in Differential Geometry (London Math. Soc. Lecture Note Ser. 124)* (Cambridge: Cambridge University Press)

[17] Malek F and Shafei Deh Abad A 1996 Homogeneous Poisson structures *Bull. Aust. Math. Soc.* **54** 203–10

[18] Marle Ch-M 1985 Quelques propriétés des variétés de Jacobi *Géométrie Symplectique et Mécanique Séminaire Sud-Rhodanien de Géométrie* ed J-P Dufour (Paris: Hermann) pp 125–39

[19] Marle Ch-M 1991 On Jacobi manifolds and Jacobi bundles *Symplectic Geometry, Groupoids and Integrable Systems Séminaire Sud-Rhodanien (Berkeley, USA, 1989)* (*Mathematical Sciences Research Institute Publications 20*) ed P Dazord and A Weinstein (Berlin: Springer) pp 227–46

[20] Marle Ch-M 1998–9 Variétés symplectiques et variétés de Poisson Cours de DEA Université Pierre et Marie Curie webpage http:// www.math.jussieu.fr/∼marle

[21] Nunes da Costa J M 1989 Réduction des variétés de Jacobi *C. R. Acad. Sci., Paris* I **308** 101–3

[22] Nunes da Costa J M and Petalidou F 2002 Reduction of Jacobi-Nijenhuis manifolds *J. Geom. Phys.* **41** 181–95

[23] Reeb G 1952 Sur certaines propriétés topologiques des trajectoires des systèmes dynamiques *Mém. Acad. R. Belg. Sci.* **27** 130–94

[24] Schouten J A 1954 On the differential operators of the first order in tensor calculus *Convegno Int. Geom. Diff. (Italia 1953)* ed Cremonese pp 1–7

[25] Stefan P 1974 Accessibility and foliations with singularities *Bull. Am. Math. Soc.* **80** 1142–5

[26] Vaisman I 1994 *Lectures on the Geometry of Poisson Manifolds (Progress in Mathematics 118)* (Basel: Birkhaüser)